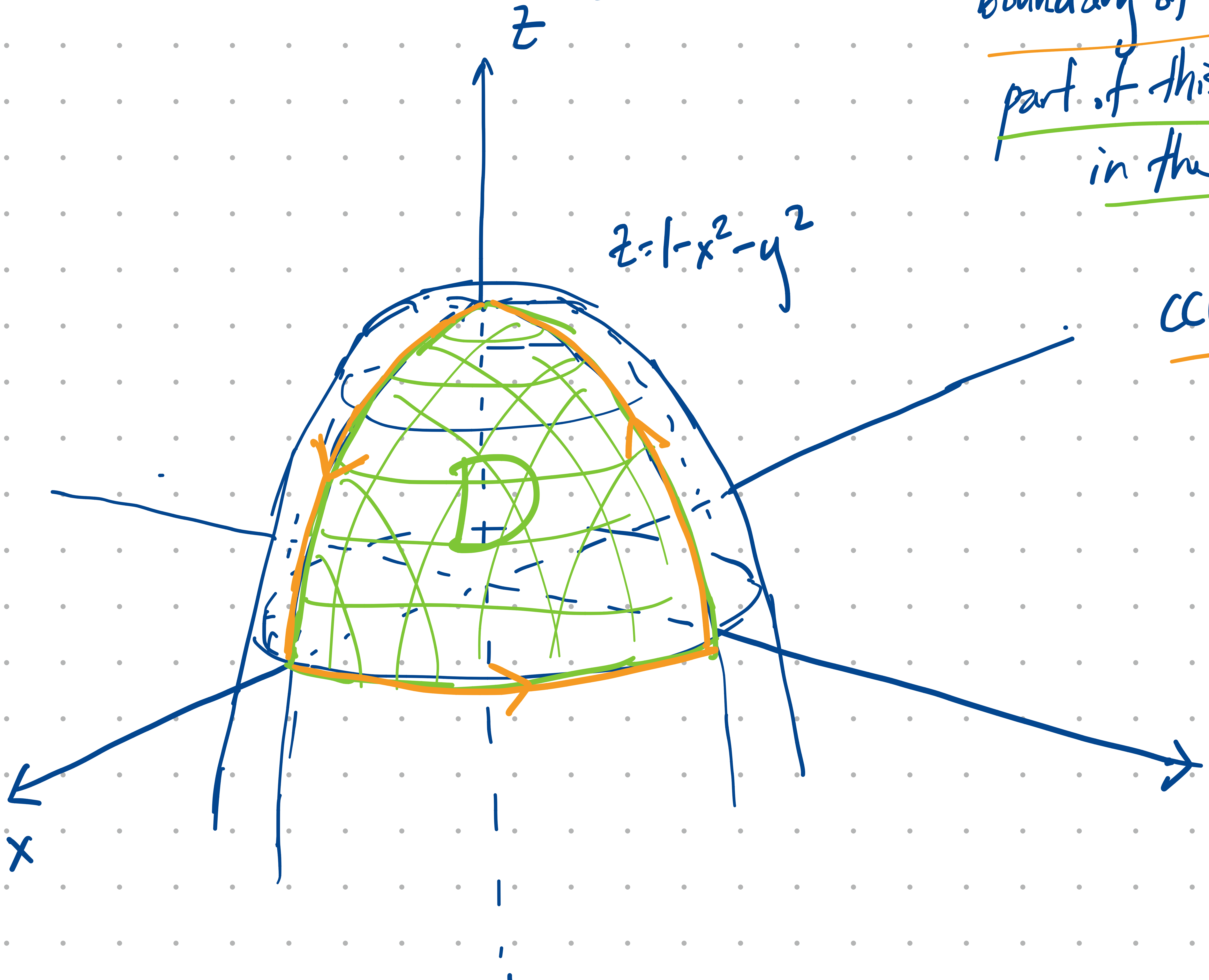
16.8 #9

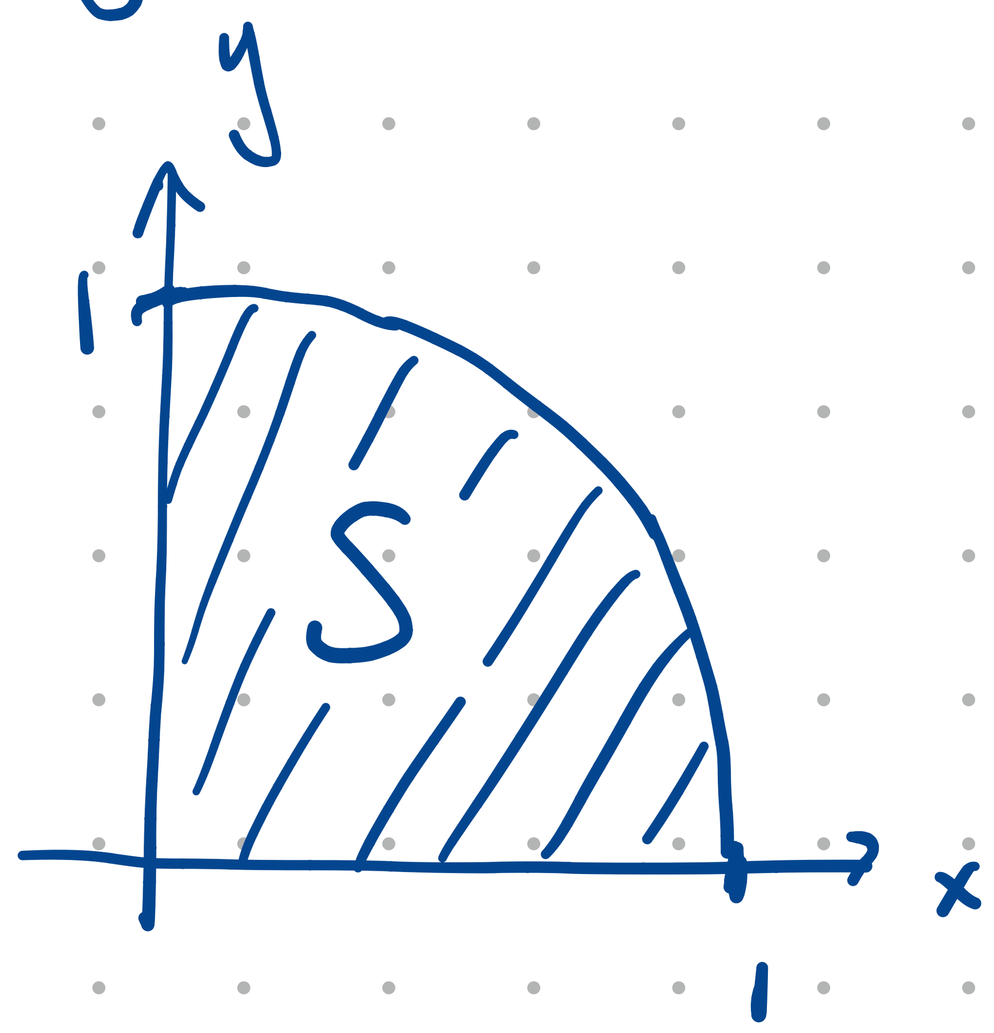
Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the boundary of the part of this paraboloid in the first octant, CCW from above



$\vec{F}=\langle xy, yz, zx\rangle$

Method 1:

$\vec{r}(x,y) = \langle x, y, 1-x^2-y^2 \rangle$



Set up

$$\iint_S \underline{(\nabla \times \vec{F}) \cdot (\vec{r}_x \times \vec{r}_y)}\, dx\, dy$$

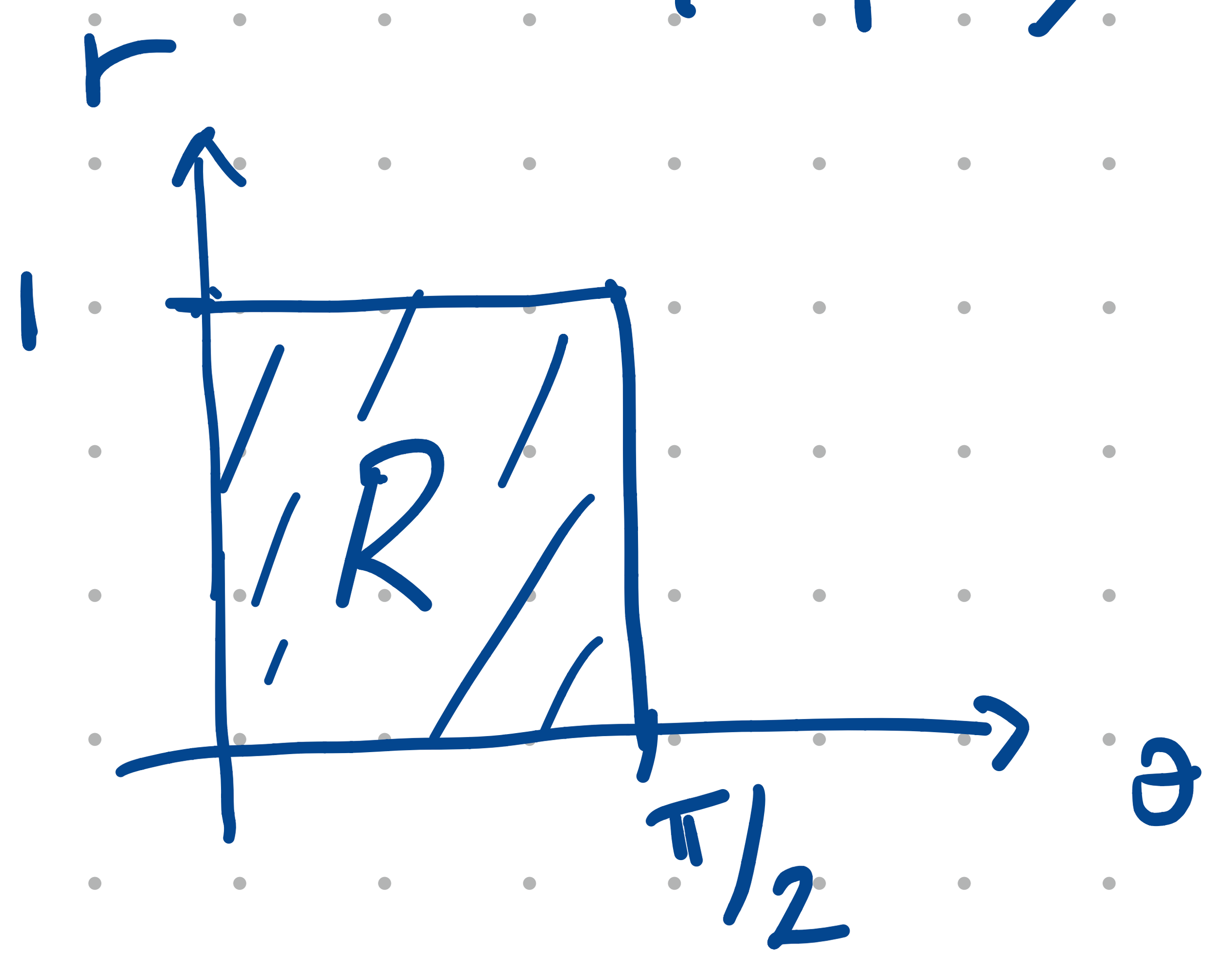
this is now a Ch 15 problem

$$\int_0^{\pi/2} \int_0^1 \quad r\, dr\, d\theta$$

integral over R

Method 2:

$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1-r^2 \rangle$



Set up:

$$\iint_R (\nabla \times \vec{F}) \cdot (\vec{r}_r \times \vec{r}_\theta)\, dr\, d\theta$$

there is <u>no</u> r here.

16.9 #24

S: sphere $x^2+y^2+z^2=1$

$$\iint_S (2x+2y+z^2)\,dS$$

Evaluate using Divergence Thm

$$\iint_{S \text{ outwards}} \vec{F}\cdot d\vec{S} = \iiint_{x^2+y^2+z^2\le 1} (\operatorname{div}\vec{F})\,dV$$

Issue: the surface integral we're given doesn't look like the left hand side above :(

Remember, flux integrals are special kinds of scalar ones:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \underbrace{(\vec{F}\cdot\hat{n})}_{\text{scalar fn.}}\,dS$$

The idea: Can we write

$$\iint_S (2x+2y+z^2)\,dS = \iint_S (\vec{F}\cdot\hat{n})\,dS$$

for some $\vec{F}$?

$$= \iint_S \vec{F}\cdot d\vec{S}$$

$\hat{n}$ = unit (outward) normal

$$= \frac{\nabla(x^2+y^2+z^2)}{\text{magnitude}} = \frac{\langle 2x, 2y, 2z\rangle}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \langle x, y, z\rangle$$ (this is outwards as desired.)

So

$$\iint_S (2x+2y+z^2)\,dS = \iint_{S \text{ outwards}} \langle 2,2,z\rangle\cdot d\vec{S} = \iiint_{x^2+y^2+z^2\le 1} 1\,dV$$

$$\operatorname{div}\langle 2,2,z\rangle = \frac{\partial}{\partial x}(2) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(z) = 0+0+1.$$

At this point could finish the problem by interpreting it as vol of sphere (which can be easily derived using methods of Ch. 15).

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§16.7 #27

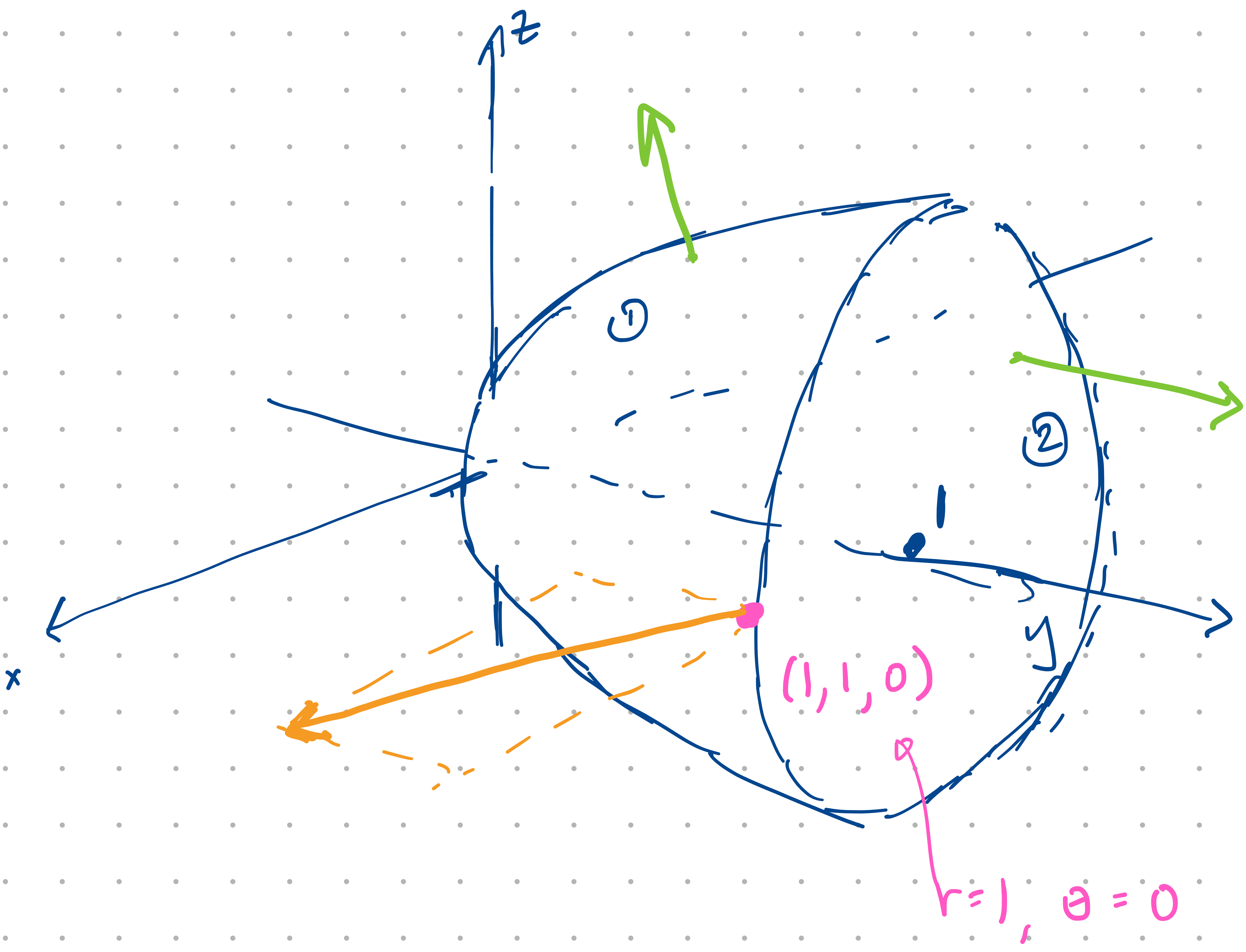
$\vec{F} = \langle 0, y, -z \rangle$

S: ① $y = x^2 + z^2 \quad 0 \le y \le 1$

together with

② $y = 1 \quad x^2 + z^2 \le 1$

oriented out



When setting up the surface integrals...

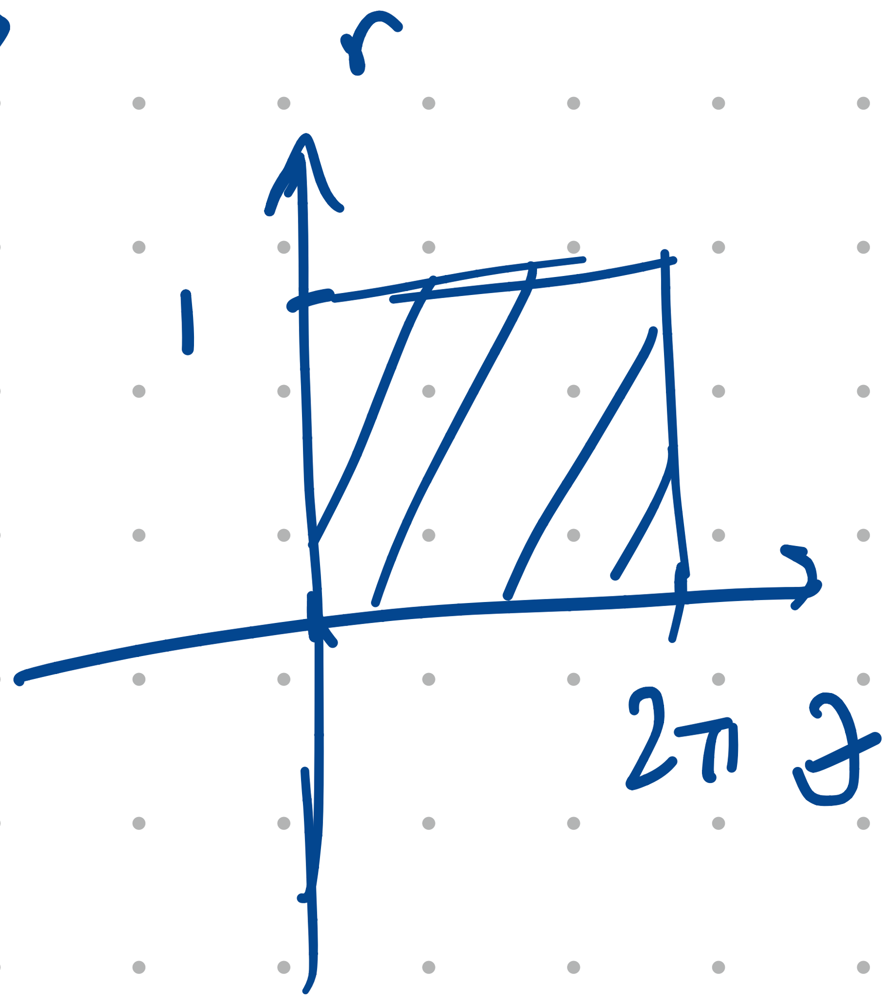
① Using "polar" param directly:

$x = r\cos\theta$
$z = r\sin\theta$

$$\vec{r}(r,\theta) = \langle r\cos\theta,\ r^2,\ r\sin\theta \rangle$$

$$\vec{r}_r = \langle \cos\theta,\ 2r,\ \sin\theta \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta,\ 0,\ r\cos\theta \rangle$$



Let's compute $\vec{r}_r \times \vec{r}_\theta$:

$$= \langle 2r^2\cos\theta,\ -r,\ 2r^2\sin\theta \rangle$$

Q: did we pick the right one?

To check, plug in convenient parameter values (that don't give $\vec{0}$)

$r = 1,\quad \theta = 0$: $\langle 2, -1, 0 \rangle$

drawing this on the picture, we see this is the desired direction.

$$\int_0^{2\pi}\int_0^1 \langle 0, r^2, -r\sin\theta\rangle \cdot \langle 2r^2\cos\theta, -r, 2r^2\sin\theta\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(-r^3 - 2r^3\sin\theta\right) dr\, d\theta$$

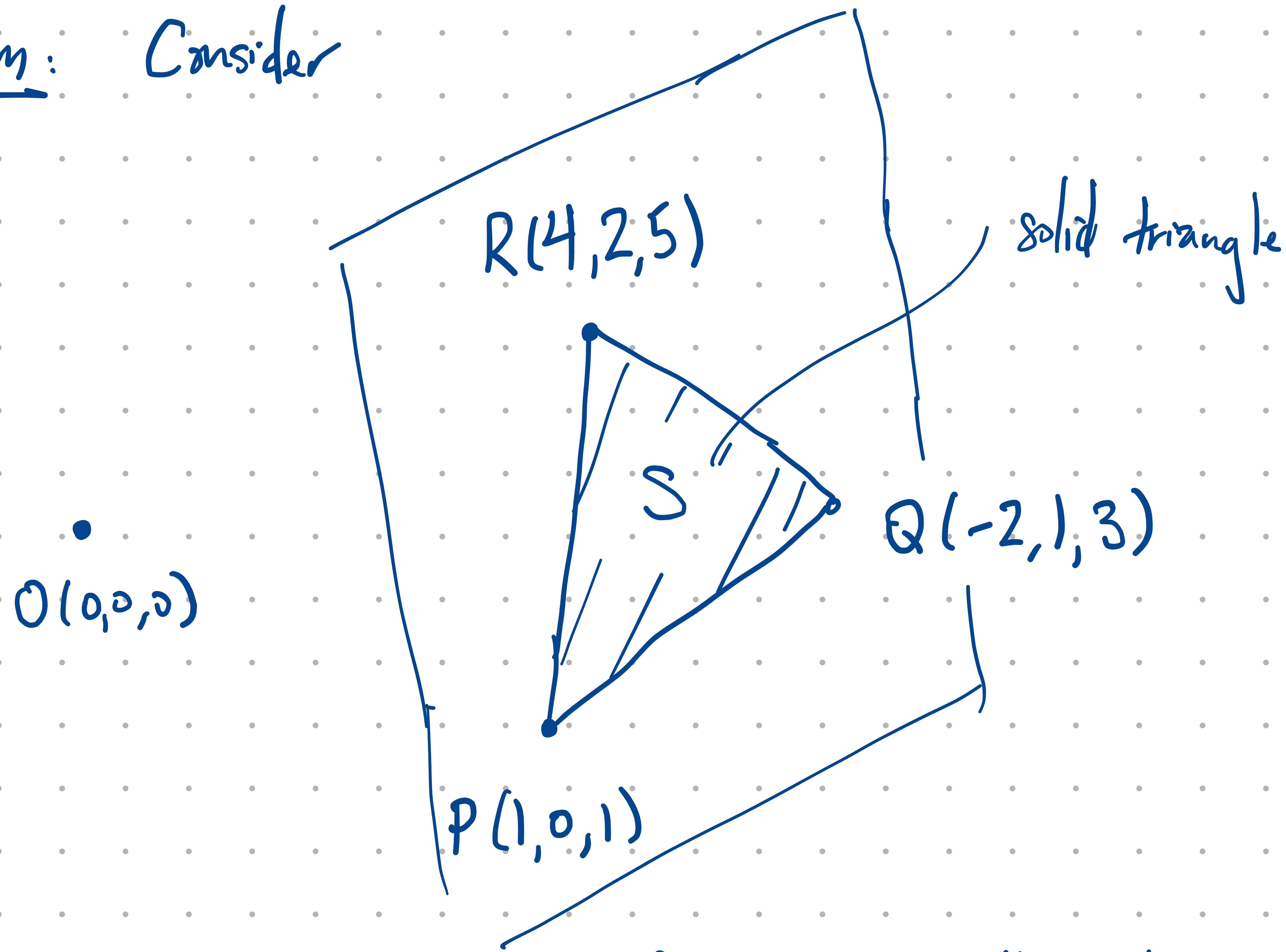
Exercise (really do it) try param. as

$$\vec{r}(x,z) = \langle x, 1-x^2-z^2, z\rangle$$

and then switching to polar, and checking you get same integral.

2) similar business for other face

Problem: Consider



Compute the flux of $\langle x, y, z\rangle$ through S in the direction away from the origin.

a) Do this by first parametrizing S, and then setting up the integral directly

b) Can you figure out a way to solve this using the Divergence Thm?